Math 53, Discussions 116 and 118 **2022-10-14**

Some Midterm 2 Review

Questions

Question 1. Evaluate the limit if it exists:

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^4+y^4}.$$

Question 2. Suppose $f(x,y)$ is a differentiable function and $g(u,v) = f(e^u + \cos v, e^u + \sin v)$. Use the values below to calculate $g_u(0,0)$ and $g_v(0,0)$.

$f(0,0) = 3$	$g(0,0) = 6$	$f_x(0,0) = 4$	$f_y(0,0) = 8$
$f(2,1) = 6$	$g(2,1) = 3$	$f_x(2,1) = 2$	$f_y(2,1) = 5$

Question 3. Let $f(x,y) = \sqrt{xy}$.

(a) Compute the gradient of f.
(b) Find the equation of the tangent plane to the graph $z = f(x,y)$ when $(x,y) = (2,8)$.
(c) Find the directional derivative of $f(x,y)$ at $P(2,8)$ in the direction towards the point $Q(5,4)$.

Question 4. Find and classify the critical points of the function

$$f(x,y) = \sin x \sin y$$

Question 5. Find and classify the critical points of the function

$$f(x,y) = x^2 - 4x - y^4.$$

Question 6. Let x, y, z denote the side lengths of a triangle. Heron's formula says that the area of the triangle is

$$A = \sqrt{s(s-x)(s-y)(s-z)}$$

where s is the semiperimeter $s = \frac{1}{2}(x+y+z)$.

Show that if s is fixed, A is maximized when $x = y = z$ (meaning that the triangle is equilateral).

Question 7. Find the absolute maxima and minima of the function $f(x,y) = xy^2$ on the region $x \geq 0, y \geq 0, x^2 + y^2 \leq 3$.

Below are brief answers to the worksheet exercises. If you would like a more detailed solution, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to questions

Question 1. Along the x-axis we get

$$\lim_{x\to 0} \frac{1}{x}$$

which evidently doesn't exist. So the original limit doesn't exist either.

Question 2. Let $x = e^u + \cos v$ and $y = e^u + \sin v$. The chain rule says

$$g_u(u,v) = f_x(x,y)\frac{\partial x}{\partial u} + f_y(x,y)\frac{\partial y}{\partial u}$$

and similarly

$$g_v(u,v) = f_x(x,y)\frac{\partial x}{\partial v} + f_y(x,y)\frac{\partial y}{\partial v}.$$

The key thing to note is that $(u,v) = (0,0)$ means $(x,y) = (2,1)$, just by how we defined x, y. So:

$$g_u(0,0) = (2)(1) + (5)(1) = 7, \qquad g_v(0,0) = (2)(0) + (5)(1) = 5.$$

Question 3.

(a) The gradient is $\nabla f(x,y) = \langle \frac{y}{2\sqrt{xy}}, \frac{x}{2\sqrt{xy}} \rangle$.

(b) The tangent plane is

$$z = 4 + (x-2) + \frac{1}{4}(y-8).$$

(c) The vector from P to Q is $\langle 3, -4\rangle$. The unit vector in this direction is $\langle 3/5, -4/5\rangle$. Hence the directional derivative is

$$\langle 1, 1/4\rangle \cdot \langle 3/5, -4/5\rangle = 2/5.$$

Question 4. The critical points are the solutions to the system of equations

$$\cos x \sin y = 0$$
$$\sin x \cos y = 0.$$

From the first equation, we see that eithre $\cos x = 0$ or $\sin y = 0$. In the former case, the second equation implies that $\cos y = 0$ (since $\sin x = \pm 1 \neq 0$). Likewise, in the latter case, the second equation implies that $\sin x = 0$. So we have two cases.

For the case $\cos x = \cos y = 0$, the points are

$$(x,y) = (\pi/2 + m\pi, \pi/2 + n\pi)$$

where $m, n \in \mathbb{Z}$. For the case $\sin x = \sin y = 0$, the points are

$$(x,y) = (m\pi, n\pi)$$

where $m, n \in \mathbb{Z}$.

Next we have to classify the points. We have

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} -\sin x \sin y & \cos x \cos y \\ \cos x \cos y & -\sin x \sin y \end{bmatrix}.$$

At the points $(x,y) = (\pi/2 + m\pi, \pi/2 + n\pi)$, we have $f_{xy} = 0$. The quantities f_{xx} and f_{yy} happen to be equal; they are either 1 or −1. The sign depends on m, n. If $m+n$ is even, then $f_{xx} = f_{yy} = -1$, so we have a local max. If $m+n$ is odd, then $f_{xx} = f_{yy} = 1$ so we have a local min.

At the points $(x,y) = (m\pi, n\pi)$, we have $f_{xx} = f_{yy} = 0$. The quantity f_{xy} is 1 or −1 depending on m, n. But in either case, $D < 0$, so we have a saddle point.

Question 5. The critical points solve the system

$$2x - 4 = 0$$
$$-4y^3 = 0$$

so $(2,0)$ is the only critical point. However, as you can check, $D = 0$ at this point so the 2nd derivative test is inconclusive. We'll have to rely on the definitions of local extrema instead.

Note that $f_{xx}(2,0) = 2 > 0$. So from the SVC 2nd derivative test, we know that along the x direction, the point $(2,0)$ is a local minimum. This does not imply that $(2,0)$ is a local minimum of $f(x,y)$, *but* it certainly eliminates the possibility that it's a local max.

However, if we consider the behavior in the y direction, i.e. $f(2,y) = -4 - y^4$, we see that as y deviates from 0 the value of f decreases. So $(2,0)$ isn't a min.

From this, we conclude that $(2,0)$ is a saddle point.

Question 6. Let $g(x,y,z) = x + y + z$ so $g(x,y,z) = 2s$ is our constraint (recall that s is a constant). We can use $f(x,y,z) = (s-x)(s-y)(s-z)$, since the inputs which maximize f would also maximize A. Lagrange tells us to solve the system

$$-(s-y)(s-z) = \lambda$$
$$-(s-x)(s-z) = \lambda$$
$$-(s-x)(s-y) = \lambda$$

i.e.

$$(s-y)(s-z) = (s-x)(s-z) = (s-x)(s-y).$$

Note that if any one of x, y, z is equal to s, then $A = 0$, which minimizes rather than maximizes the quantity of interest. So We can assume that $s-x, s-y, s-z$ are all nonzero. We conclude from the preceding equations that $x = y = z$, as desired. (They are all equal to $2s/3$.)

Question 7. We compile a list of candidates for extrema. The corners $(0,0)$, $(\sqrt{3},0)$, $(0,\sqrt{3})$ are candidates.

For the interior $x > 0, y > 0, x^2 + y^2 < 3$ (strict inequalities) the candidates are critical points. The critical points of f lie along the x-axis, so there are none in the region of interest.

Next we have the three bounding edges. We see that the value of f is zero along all of the points along the x and y axes, so it remains to consider the circular part.

This can be handled with e.g. Lagrange multipliers:

$$x^2 + y^2 = 3$$
$$y^2 = \lambda 2x$$
$$2xy = \lambda 2y.$$

The last equation implies either $y = 0$ or $x = \lambda$. But the arc in question has only points $y > 0$, so we eliminate the first case. Hence $y \neq 0$ and $x = \lambda$, meaning $y^2 = 2x^2$ from the second equation. Plugging into the first yields $3x^2 = 3$, so $x = \pm 1$, but only $x = 1$ is relevant. Thus $y = \sqrt{2}$.

Altogether, we conclude that the maxima of f are attained along the axes, where $f = 0$, and the maximum of f is $f(1,\sqrt{2}) = 2$.